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A study of the temperature dependence of the elementary excitations in Helium II by the use of cold neutrons

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With 10 figures in the text

ABSTRACT

Recently the dispersion curve of the elementary excitations in Helium II has been studied by observing the angular and energy distribution of cold neutrons scattered from liquid helium, in Stockholm [1, 2] in the momentum region $1.50 \leqslant \varkappa \leqslant 2.14$ Å^{-1} at a helium temperature of 1.44°K, and at Los Alamos [3] in the region $0.55 \leqslant \varkappa \leqslant 2.36$ Å^{-1} at a helium temperature of about 1.1°K. The latter workers also studied the variation with helium temperature up to 1.8°K of parameters of the dispersion curve such as the width of the roton line and the minimum excitation energy Δ in the roton region, where the dispersion curve according to Landau is given by

$$E = \Delta + \frac{(p - p_0)^2}{2\mu}$$

Δ is the minimum energy to create a roton, p the momentum of the roton and μ its effective mass. Independently of the Los Alamos work the variation with temperature of the dispersion curve has been studied at the Stockholm reactor in the temperature region $1.44 \leqslant T \leqslant 2.37$°K. The observed temperature dependence of Δ is very well fitted by the functions

$$\frac{\Delta}{k_B} = \begin{cases} 8.5\left(1 - 7.5 \cdot \sqrt{T} \cdot e^{-\frac{8.5}{T}}\right) & \text{for } 1.4 \leqslant T \leqslant 1.9^\circ\text{K} \\ 8.5\left\{1 - 1050 \cdot \left(\sqrt{T} \cdot e^{-\frac{8.5}{T}}\right)^{13/6}\right\} & \text{for } 1.8 \leqslant T \leqslant 2.15^\circ\text{K} \end{cases}$$

The experimentally observed width δE of the roton line is well described by

$$\frac{\delta E}{k_B} = \begin{cases} 302 \cdot \sqrt{T} e^{-\frac{8.5}{T}} & \text{for } T < 2^\circ\text{K} \\ 2060 \cdot \left(\sqrt{T} e^{-\frac{8.5}{T}}\right)^{3/2} & \text{for } T \geqslant 2^\circ\text{K} \end{cases}$$

Introduction

Shortly after the theoretical prediction by Feynman and Cohen [4] that cold neutrons could be used to excite single excitations in liquid helium, experiments [1, 2, 3] were performed which confirmed this idea. Moreover, these experiments successfully showed that the basic concepts of Landau [5], Feynman [6], and Brueckner and Sawada [7] were correct: the energy vs. momentum relation for the excitations showing a marked minimum at a momentum of about 1.90 Å^{-1}, the minimum energy $\frac{\Delta}{k_B}$, necessary to create a roton being about 8.1–8.6°K at helium temperatures between 1.5 and 1.1°K. Theoretically wery little is known about what happens at higher helium temperatures, up to the λ-point, when the density of the excitations increases fast. According to the Landau-Khalatnikov theory the roton concentration, N_r (number of rotons per unit volume) should increase as $\sqrt{T} \cdot e^{-\frac{\Delta}{T}}$ with helium temperature. This would result in a decreasing mean life time of the rotons and consequently decreasing mean free path. This in turn would correspond to a roton line width increasing with temperature as $\sqrt{T} \cdot e^{-\frac{\Delta}{T}}$. This law would then be valid up to helium temperatures of about 1.8°K. In the neighbourhood of the λ-point there is no theoretical prediction. Also there is no theory for any variation of the "gap" energy Δ with helium temperature, except the attempts of Brueckner and Sawada to calculate the dispersion curve for a dense Bose gas. For these reasons we undertook a systematic study of the temperature variation of the dispersion curve using the cold neutron technique.

Experiments

a) Experimental method and resolution of neutron spectrometer

In earlier papers [1, 2] the principles used to produce single excitations in Helium II by cold neutrons have been described in detail. In the present investigation the same apparatus [8] and the same technique of measurement have been used. The angular and energy distribution of the beryllium filtered neutron beam scattered from a liquid helium sample is analyzed by means of a slow chopper time-of-flight spectrometer. The neutron spectrum impinging upon the helium sample is characterized by a sharp edge at a neutron wave length of 3.95 Å corresponding to the last Bragg cut-off of beryllium. The resolution width of the spectrometer is such that the full width, $(\delta t)_0$, at half maximum of the resolution function, assumed gaussian in shape, is $100 \pm 5\,\mu$sec, when the time-of-flight of 4 Å neutrons is about 3000 μsec. The primary spectrum scattered elastically from a vanadium sample is given in Fig. 1, where the resolution width is shown as well as the position of the beryllium break. The beryllium cut-off very nearly corresponds to the position of the point of inflection of the sharp edge, which is rounded off by the finite resolution function. The determination of the true break position is slightly complicated by the presence of an aluminium "break" at 4.05 Å. Another aluminium "break" shows up at a wave length of 4.65 Å (compare Fig. 1 and reference 8). This complication arises from constructional aluminium in the reactor tank and experimental channel used. Calculation shows, however, that the correction due to this cause is small, amounting to about 3 μsec in 3000 μsec. Another complication arises from the fact that the filtering action of the beryllium filter of 20 cm thickness is incomplete, resulting in a minor transmission at the 3.58 Å beryllium "break" corresponding to the peak at a flight time 2850 μsec in Fig. 1. This fact also causes a minor

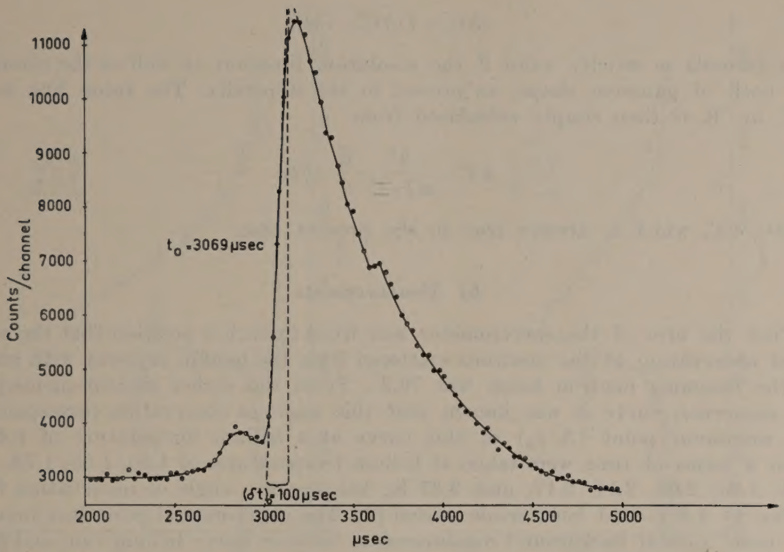

Fig. 1. Spectrum of beryllium filtered neutrons as observed by scattering from vanadium. The half width value of the resolution function is given as the projection of the tangent at the point of inflection on the time-of-flight axis and is found to be 100 μsec.

uncertainty of the order 2 or 3 μsec in the definition of the 3.95 Å cut-off. The corrected value for the time-of-flight (t_0) of the 3.95 Å neutrons used in the treatment of the present data is 3069 ± 5 μsec. The accurate calculation of the shape of a triangular neutron spectrum as seen after the application of a gaussian resolution function and a discussion of the corrections is given in the appendix.

If the flight time of the neutrons after the scattering process in the helium sample is denoted by t_f the laws of energy and momentum concervation give for the excitation energy ($T°K$) and the corresponding momentum value ($\varkappa$ Å^{-1}):

$$T = \frac{h^2}{2 m k_B \lambda_c^2} \cdot \frac{t_0 + t_f}{t_f^2} \cdot \Delta t \tag{1}$$

$$\varkappa = \frac{2\pi}{\lambda_c} \cdot \frac{\sqrt{t_0^2 + t_f^2 - 2 t_0 t_f \cos \theta}}{t_f}, \tag{2}$$

where m is the neutron mass, k_B is the Boltzmann constant, λ_c is the beryllium cut-off (3.952 Å at liquid nitrogen temperature), θ is the angle of observation, and $\Delta t = t_f - t_0$ is proportional to the wave length shift caused by the scattering process (compare reference 2).

If any change in the slope of the scattered beryllium cut-off is observed, such that the break appears symmetric but of a width $(\delta t)_f$, the true roton line width $(\delta t)_r$, expressed in time units, may be calculated from

$$(\delta t)_r = \sqrt{(\delta t)_f^2 - (\delta t)_0^2} \tag{3}$$

This formula is strictly valid if the resolution function as well as the roton line are both of gaussian shape, as proved in the appendix. The roton line width, ΔT, in °K is then simply calculated from

$$\delta T = \frac{h^2}{m k_B \lambda_c^2} \cdot \frac{t_0^2}{t_f^3} \cdot (\delta t)_r \tag{4}$$

if $(\delta t)_r \ll t_f$, which is always true in the present case.

b) Measurements

First the arm of the spectrometer was fixed in such a position that the angle, θ, of observation of the neutrons scattered from the helium cryostat with respect to the incoming neutron beam was 76.3°. From the earlier measurements [2] of the dispersion curve it was known that this angle of observation corresponds to the minimum point $(\Delta, \varkappa_0)$ of this curve at a helium temperature of 1.44°K. Then a series of runs were taken at helium temperatures of 1.50, 1.65, 1.73, 1.85, 1.94, 1.98, 2.03, 2.14, 2.17, and 2.37°K, keeping the angle of observation fixed. A run at 4.2°K had been made earlier [2]. The experimental procedure involved the usual careful background measurements between every helium run, and check of the calibration of the time scale between the helium runs using the elastic scattering from vanadium. As demonstrated in Fig. 2 two effects were observed as the helium temperature was increased:

1. The slope of the scattered beryllium break, characterized by $(\delta t)_f$, increased, corresponding to an increasing roton line width.
2. The position of the break, t_f, shifted continuously towards shorter flight times, corresponding to an apparently decreasing value of the excitation energy.

The question then arises whether the excitation energies which can be calculated from these series of observations really correspond to the minimum, Δ, of the dispersion curve. To investigate the shape of the dispersion curve in the neighbourhood of the minimum a series of observations at different angles was therefore made at a helium temperature of 2.03°K. The results of these measurements are given in Table 1. The values of the excitation energies (T°K) and the corresponding momenta ($\varkappa$ Å^{-1}) have been calculated using equations (1) and (2), taking $t_0 = 3069 \pm 5\,\mu$sec.

Table 1. Dispersion curve determined at a helium temperature of 2.03°K.

Angle of observation θ	Time-of-flight for scattered neutrons t_f μsec	Time shift of break position Δt μsec	Excitation energy T°K	Momentum value $\varkappa$ Å^{-1}
90.0	3320 ± 10	251 ± 12	8.8 ± 0.4	2.16 ± 0.5
83.0	3266 ± 10	197 ± 12	7.1 ± 0.4	2.04 ± 0.5
80.0	3255 ± 10	186 ± 12	6.7 ± 0.4	1.98 ± 0.5
76.3	3244 ± 10	175 ± 12	6.4 ± 0.4	1.91 ± 0.5
74.0	3263 ± 15	194 ± 16	7.0 ± 0.5	1.86 ± 0.5
71.5	3278 ± 15	209 ± 16	7.5 ± 0.5	1.80 ± 0.5

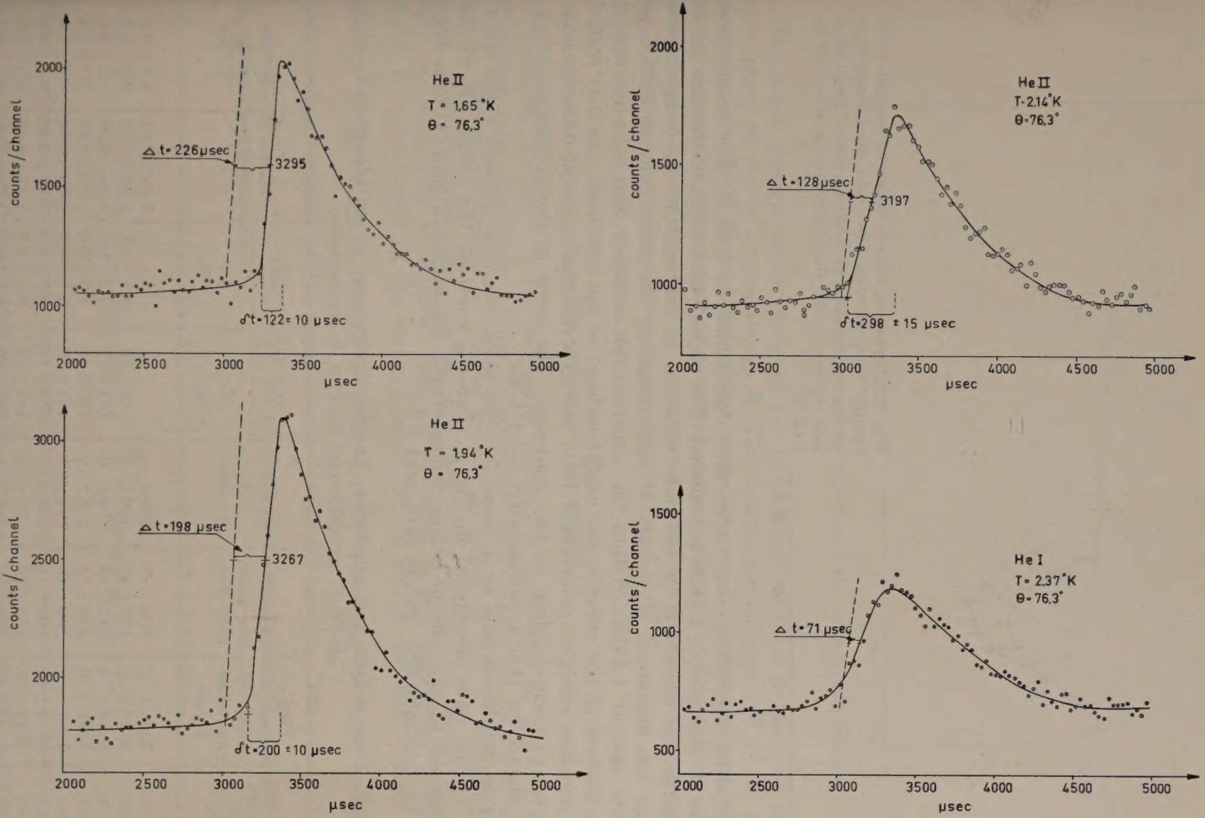

Fig. 2. Examples of spectra of neutrons scattered from liquid helium at various sample temperatures. The time shift, Δt, of the spectrum caused by the scattering process as well as the observed half width, δt, of the break are shown in the figure.

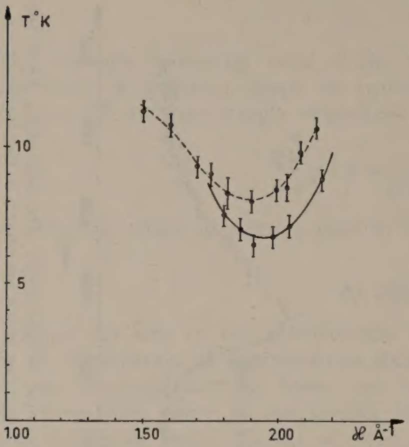

Fig. 3. Dispersion curves taken at two different sample temperatures. The solid line corresponds to the curve taken at 2.03°K, while the broken line represents the dispersion curve observed at 1.44°K.

Fig. 3 shows the dispersion curve thus determined at 2.03°K. For comparison the curve observed at 1.44°K is included. The important feature of the new curve as regards the discussion of the data obtained at different helium temperatures is, that the minimum, $\varkappa_0$, occurs at approximately the same value at 2.03°K as was the case at 1.44°K. Within the experimental accuracy, and with due regard to the flatness of the curve in the neighbourhood of the minimum, one may regard the excitation energies determined from the experimental series at different temperatures as corresponding to the minimum. The error in the excitation energy, T°K, caused by this approximation is $\leqslant 0.1$°K.

Having establised this as a basis for discussion, we now proceed to the treatment of the data illustrated in Fig. 2. Table 2 contains the results of all the observations at the various helium temperatures. Equations (3) and (4) have been used to calculate line widths $(\delta t)_r$ and δT.

Table 2. Line widths and excitation enrgies determined at an observation angle of 76.3° at different helium temperatures.

Helium temperature °K	Flight time of scattered neutrons t_f µsec	Observed line width $(\delta t)_f$ µsec	Line shift Δt µsec	Roton line width		Roton momentum $\varkappa$ Å^{-1}	Roton energy T °K
				$(\delta t)_r$ µsec	δT °K		
1.50	3305 ± 6	100 ± 7	236 ± 8			1.89 ± 0.04	8.3 ± 0.4
1.65	3295 ± 6	122 ± 10	226 ± 8	70 ± 12	2.2 ± 0.8	1.90 ± 0.04	8.0 ± 0.4
1.73	3289 ± 6	142 ± 10	220 ± 8	101 ± 12	3.2 ± 0.6	1.90 ± 0.04	7.8 ± 0.4
1.85	3285 ± 6	158 ± 10	216 ± 8	122 ± 12	3.9 ± 0.6	1.90 ± 0.04	7.7 ± 0.4
1.94	3267 ± 6	200 ± 10	198 ± 8	173 ± 12	5.7 ± 0.5	1.90 ± 0.04	7.1 ± 0.4
1.98	3259 ± 10	200 ± 15	190 ± 12	173 ± 16	5.7 ± 0.7	1.91 ± 0.05	6.9 ± 0.5
2.03	3244 ± 10	213 ± 15	175 ± 12	188 ± 16	6.3 ± 0.7	1.91 ± 0.05	6.4 ± 0.5
2.14	3197 ± 10	298 ± 15	128 ± 12	281 ± 16	9.8 ± 0.7	1.92 ± 0.05	4.8 ± 0.6
2.17	3157 ± 30	300 ± 30	88 ± 30	283 ± 31	10.3 ± 1.4	1.93 ± 0.06	3.3 ± 1.3
2.37	(3140 ± 15)	(330 ± 30)	(71 ± 16)	(314 ± 31)	(11.6 ± 1.3)		(2.7 ± 0.8)

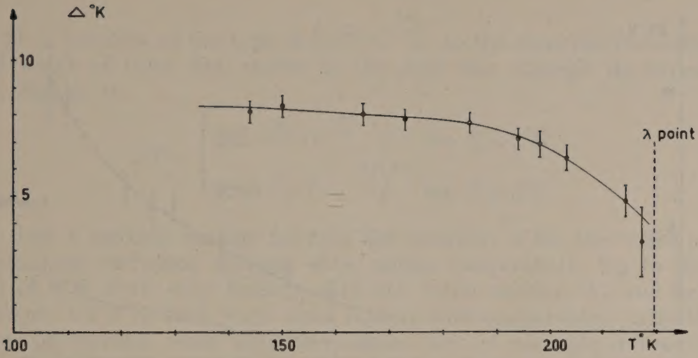

Fig. 4. Variation of "gap" energy, Δ, as a function of the helium temperature.

The roton energy values given in the last column of Table 2 thus very nearly correspond to the minimum energy Δ. This assumption is supported by the roton momenta belonging to the roton energy values as seen in the last-but-one column, which shows that the momentum values increase slightly with increasing helium temperature. But this is exactly what may be expected for the minimum according to the two complete measurements at 1.44 and 2.03°K.

Discussion of the results

According to Table 2 the minimum energy, Δ, necessary to create a roton decreases with increasing helium temperature first slowly up to about 1.8 or 1.9°K, then more rapidly up to the λ-point as shown in Fig. 4. Assuming that some unknown interaction between the excitations causes this change in Δ, it may be reasonable to suppose that the strength of this interaction is proportional to some power of the concentration, N_r, of rotons per unit volume. According to the Landau theory, $N_r \propto \sqrt{T} \cdot e^{-\frac{\Delta}{k_B T}}$. Therefore we tried to fit our data with a function of the form $\Delta = A \cdot \left\{1 - B\left(\sqrt{T} \cdot e^{-\frac{c}{T}}\right)^n\right\}$. The result of such a calculation is that a very good fit to the experimental data is obtained with the following expressions:

$$\frac{\Delta}{k_B} = \begin{cases} 8.5\left\{1 - 7.5 \cdot \sqrt{T} \cdot e^{-\frac{8.5}{T}}\right\} & \text{for } 1.4 \leqslant T \leqslant 1.9^\circ\text{K} \quad (5) \\ 8.5\left\{1 - 1050 \cdot \left(\sqrt{T} \cdot e^{-\frac{8.5}{T}}\right)^{13/6}\right\} & \text{for } 1.8 \leqslant T \leqslant 2.15^\circ K \quad (6) \end{cases}$$

These expressions imply that the limiting value of $\frac{\Delta}{k_B}$ for very low temperatures is 8.5°K, and that this value decreases linearly with N_r as the temperature increases up to about 1.9°K. From this temperature and up to the λ-point the temperature dependence is stronger and the correction term goes more like the

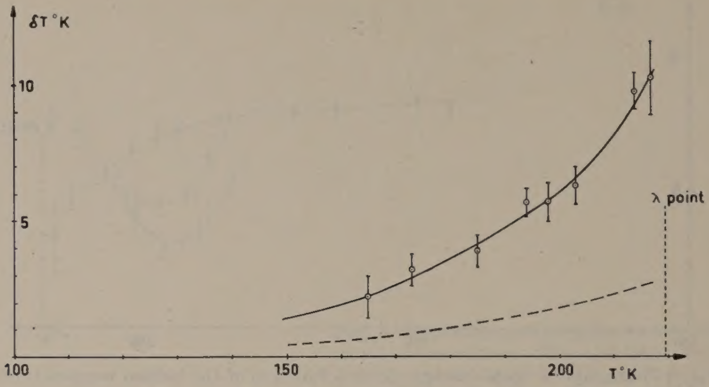

Fig. 5. Excitation line width as a function of helium temperature. The solid line corresponds to the fitted expressions (9) given in the text, while the broken line corresponds to the line width predicted by the Landau-Khalatnikov theory.

square, N_r^2, of the roton concentration. In the immediate neighbourhood of the λ-point perhaps even higher powers of N_r should be used. When the λ-point is passed the scattered neutron distribution is represented by a wide gaussian distribution (more correctly: the derivative of the scattered beryllium break is gaussian), the energy shift staying constant at a value of 2.7°K.

The line width as given in the sixth column of Table 2 is represented graphically in Fig. 5. According to the Landau-Khalatnikov theory, which assumes a constant Δ, the line width at least up to a helium temperature of 1.8°K may be calculated from

$$\delta T = \frac{\hbar T \varrho_{nr}}{5 \mu \eta_r} = \frac{2}{15} \frac{p_0^4 \cdot \sqrt{T}}{(2\pi)^{3/2} \eta_r \hbar^2 k_B \mu} e^{-\frac{\Delta}{k_B T}} \tag{7}$$

where μ is the effective roton mass, η_r is the roton part of the viscosity, ϱ_{nr} is the roton part of the normal fluid density. Taking for $\frac{\Delta}{k_B}$ the value at $T = 0$°K of 8.5°K, $\mu = 0.16\ \mathrm{m_{He}}$ (see reference 2), $\frac{p_0}{\hbar} = 1.90\ \text{Å}^{-1}$ and $\eta_r = 1.1 \cdot 10^{-5}$ poise we find

$$\delta T \simeq 92 \cdot \sqrt{T} \cdot e^{-\frac{8.5}{T}} \tag{7'}$$

This calculated line width is shown as the lower curve of Fig. 5, the calculated values being about $\frac{1}{3}$ of the experimentally observed ones. If one tries to fit an expression of the type $\delta T = A \cdot \sqrt{T} \cdot e^{-\frac{8.5}{T}}$, using the asymptotic value of 8.5°K for Δ, one finds a good fit for $A = 302$ up to a helium temperature of 2°K. From that temperature and up towards the λ-point the line width increases more rapidly. In accordance with the ideas discussed above, and assuming that some stronger interactions occur when the roton density increases to high values, we tried to fit a function of the type $A\left(\sqrt{T}\cdot e^{-\frac{8.5}{T}}\right)^{n}$ to the observed line-width data. The final result of these fits, shown as the solid line through the experimental points in Fig. 5, is:

$$\delta T = \begin{cases} 302\cdot\sqrt{T}\cdot e^{-\frac{8.5}{T}} & \text{for } T<2^{\circ}\text{K} \\ 2060\cdot\left(\sqrt{T}\cdot e^{-\frac{8.5}{T}}\right)^{3/2} & \text{for } T\geqslant 2^{\circ}\text{K} \end{cases} \tag{8}$$

There is thus a striking analogy between the variation of the line width and that of the minimum excitation energies with helium temperature. Up to the neighbourhood of 2°K both vary linearly with the roton density, N_r, and for temperatures above 1.8–2°K both vary more rapidly with temperature, indicating that some new phenomena come into play. Some sort of multiple interactions between the neutron and two or more rotons simultaneously might perhaps not be excluded, a process analogous to multiphonon scattering of neutrons from crystals.

The most striking result of the line-width determination is thus that the experimental values are about three times as large as the theoretically predicted ones. A consequence of this is, of course, that the mean free path of the rotons is three times shorter than the value calculated from the Landau-Khalatnikov theory. This discrepancy is too large to be explained by experimental errors, unless the method as a whole is masking large effects which should be interpreted differently. One such possibility would be that the roton line is asymmetric towards shorter wavelengths, resulting in too large apparent line widths and perhaps even too large line shifts. Within the given experimental resolution, however, no asymmetry of the line could be detected. Such a possibility can only be investigated using a higher resolution and ideally by sending a narrow neutron "line" into the helium sample.

Finally there is one point to be mentioned about the dispersion curve. An inspection of Fig. 3 shows that there is a slight shift of the minimum towards higher momentum values simultaneously as the minimum energy Δ decreases for increasing helium temperature. A fit of a parabola to the excitation curve observed at 2.03°K gives for the Landau parameters Δ, μ and $\varkappa_0$:

$$\begin{cases} \dfrac{\Delta}{k_B} = (6.7\pm 0.3)\ ^{\circ}\text{K} \\ \varkappa_0 = (1.94\pm 0.02)\ \text{Å}^{-1} \\ \mu = (0.13\pm 0.02)\ \text{m}_{\text{He}} \end{cases} \tag{9}$$

A comparison of these results with the previous [2] data observed at 1.44°K $\left(\frac{\Delta}{k_B}=8.1^{\circ}\text{K},\ \varkappa_0=1.90\ \text{Å}^{-1},\ \mu=0.16\ \text{m}_{\text{He}}\right)$ shows, that the tendency for the dispersion curves when the helium temperature increases is, that the curves shift towards lower excitation energies, higher momentum values, and that the concavity of the parabolic region round the minimum increases. It should be pointed out that the previous [2] measurements at 1.44°K indeed showed that the parabola shape of the spectrum does not seem to hold except in a very narrow region, $\varkappa_0\pm 0.2$ Å^{-1}, round the minimum. This statement is confirmed by the fact that the data taken at 1.44°K are very well fitted by a function of the type

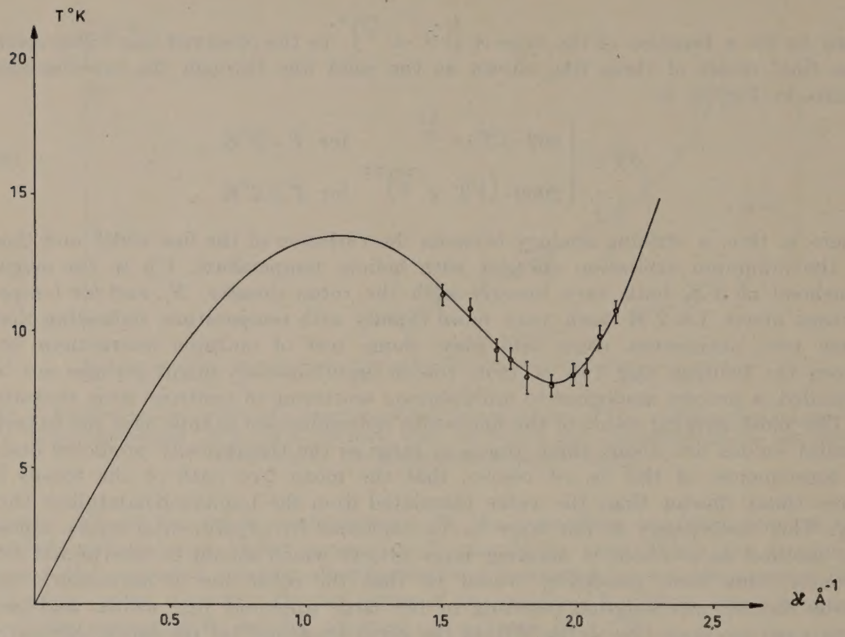

Fig. 6. Brueckner type dispersion curve fitted to the experimental data taken at a helium temperature of 1.44°K.

$$T = c_1 \varkappa \sqrt{\varkappa^2 + 2\left(\frac{c_2}{c_3}\right)^2 \cdot \frac{\sin c_3 \varkappa}{c_3 \varkappa}} \tag{10}$$

When $c_1 = 4.74 \pm 0.10$, $c_2 = 5.79 \pm 0.06$, $c_3 = 2.047 \pm 0.010$. The result of such a fit is given in Fig. 6. The expression (10) is similar to the dispersion law given by Brueckner and Sawada except that a scaling factor included in c_1 has been applied. It might be worth mentioning that this fitted curve predicts a maximum at $T = 13.53$°K and $\varkappa = 1.11$ Å^{-1}, and that the slope of this curve at the origin is 19°K per Å^{-1}, corresponding to a predicted velocity of first sound of 248 metres/sec. This value thus surprisingly well agrees with the experimental value of 237 metres/sec. The full dispersion curve thus predicted agrees relatively well with the curve experimentally observed by the Los Alamos group [3], if the temperature variation is taken into account.

APPENDIX

A gaussian resolution function applied to a step function type spectrum

Assume we have a gaussian resolution function of the type $R(t-t')$ and a neutron spectrum described by $f(t)$. t is the neutron time of flight and t' de-

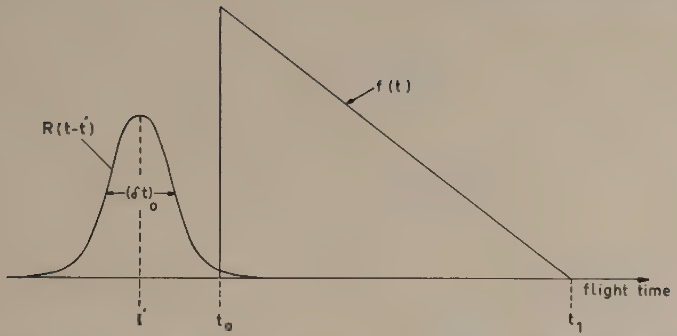

Fig. 1 (Appendix). Gaussian resolution function $R(t-t')$ applied to a triangular step function type spectrum.

fines the centre line of the resolution function (Fig. 1). The intensity $I(t')$ observed at flight time t' obviously is:

$$I(t')=\int_{-\infty}^{+\infty} R(t-t')\, f(t)\, dt, \tag{1}$$

where $R(t-t')$ is normalized in such a way that

$$\int_{-\infty}^{+\infty} R(t-t')\, dt=1 \tag{2}$$

If we take for $R(t-t')$ the function $c\cdot e^{-\frac{(t-t')^2}{a^2}}$ normalization gives:

$$R(t-t')=\frac{1}{a\sqrt{\pi}}\, e^{-\frac{(t-t')^2}{a^2}} \tag{3}$$

with

$$a=\frac{(\delta t)_0}{2\sqrt{\ln 2}} \tag{4}$$

$(\delta t)_0$ is the full width at half maximum of the resolution function R.

With these definitions we first calculate the shape of the observed spectrum for a step function neutron spectrum:

$$f(t)=\begin{cases} 0 & \text{for } t<t_0 \\ 1 & \text{for } t\geqslant t_0 \end{cases} \tag{5}$$

The result according to eq. (1) for the observed intensity is then simply:

$$I(t')=\frac{1}{a\sqrt{\pi}}\int_{t_0}^{\infty} e^{-\frac{(t-t')^2}{a^2}}\, dt \tag{6}$$

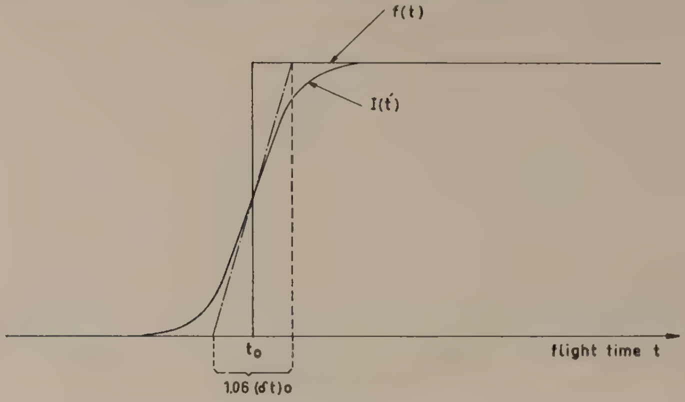

Fig. 2 (Appendix). Spectrum $I(t')$ resulting from the application of a gaussian resolution function to the step function spectrum $f(t)$. The half width of the resolution function, $(\delta t)_0$, is very nearly given by the projection of the tangent at point of inflection on the t axis. The point of inflection occurs exactly at t_0.

This observed spectrum is illustrated in Fig. 2. As is easily proved, the projection on the t axis of the tangent at the point of inflection $I(t')$ defines a time interval which is

$$a\sqrt{\pi} = \frac{(\delta t)_0}{2}\sqrt{\frac{\pi}{\ln 2}} \cong 1.06\,(\delta t)_0 \tag{7}$$

For this case we thus find that the point of inflection of $I(t')$ defines the "cut-off" t_0 exactly and the projection of the tangent on the t axis very nearly defines the full width $(\delta t)_0$ at half maximum of the resolution function.

Next we suppose the spectrum $f(t)$ is triangular as in Fig. 1 and is given by:

$$f(t) = \begin{cases} -\dfrac{t - t_1}{t_1 - t_0} & \text{for } t_0 \leqslant t \leqslant t_1 \\ 0 & \text{for } t < t_0 \text{ and } t > t_1 \end{cases} \tag{8}$$

The integration according to eq. (1) then results in intensity distributions $I(t')$ of the shape illustrated in Fig. 3. These curves are characterized by two facts:

a) The projection on the t axis of the tangent at the point of inflection still very well defines the resolution width $(\delta t)_0$ and particularly so when $(\delta t)_0 \ll t_1 - t_0$.

b) The inflection point no longer corresponds exactly to t_0 but is shifted towards a smaller t value, t_0' as shown in Fig. 3. This error in the definition of t_0, denoted by Δt in Fig. 3, increases as $t_1 - t_0$ decreases for a constant $(\delta t)_0$ but remains small as long as $(\delta t)_0 \ll t_1 - t_0$. Fig. 4 gives the error Δt as a function of $\frac{t_1 - t_0}{(\delta t)_0}$. As will be seen from this figure the correction in the actual case, when the spectrum $f(t)$ is well approximated by eq. (8), with $(\delta t)_0 \approx 100$ μsec and $t_1 - t_0 \approx 1000$ μsec, is only $3 - 4$ μsec.

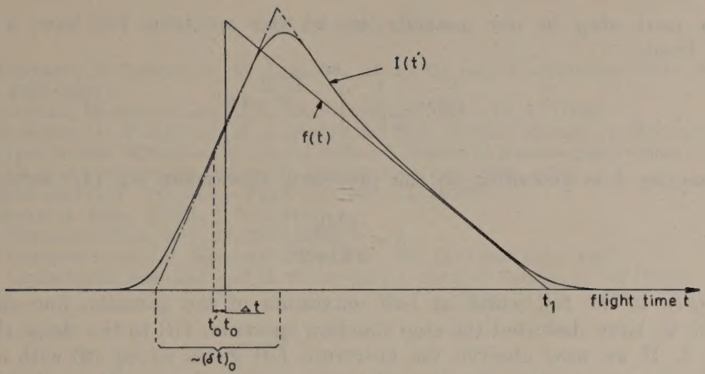

Fig. 3 (Appendix). Spectrum $I(t')$ resulting from the application of a gaussian resolution function to the triangular spectrum $f(t)$. The observed "break" position, t'_0, is now shifted by an amount Δt from the true "break" position.

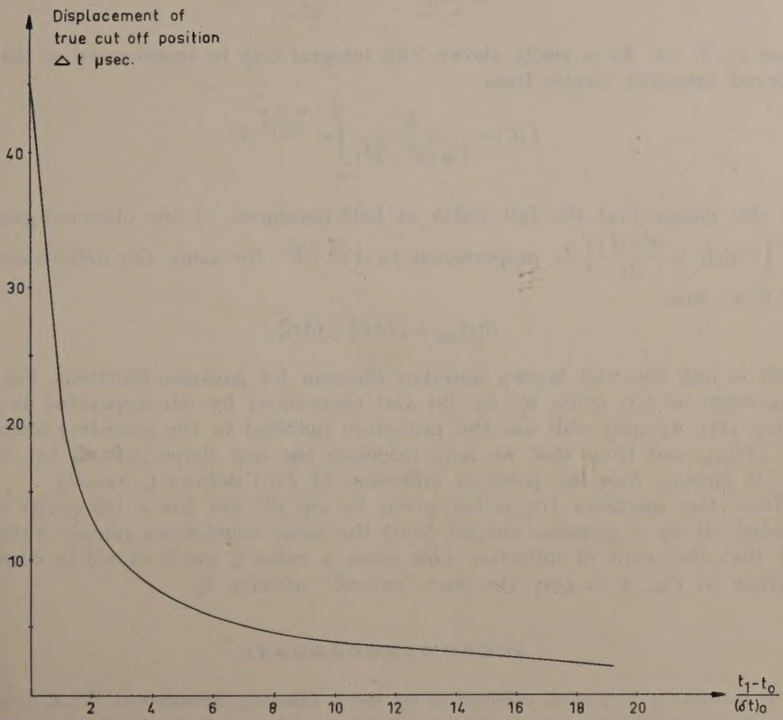

Fig. 4 (Appendix). The correction, Δt, to be added to the observed "break" position as a function of the ratio $\frac{t_1-t_0}{(\delta t)_0}$.

As a next step in our analysis we let our spectrum $f(t)$ have a gaussian shaped front:

$$f(t)=\frac{1}{b\sqrt{\pi}}\int_{t_0}^{\infty} e^{-\frac{(t''-t)^2}{b^2}}\,dt'' \tag{9}$$

The quantity b is according to the preceding discussion, eq. (4), defined by

$$b=\frac{(\delta t)_1}{2\sqrt{\ln 2}}, \tag{4'}$$

where $(\delta t)_1$ is the full width at half maximum of the gaussian line shape now assumed to have distorted the step function spectrum $f(t)$ to the shape illustrated in Fig. 2. If we now observe the spectrum $f(t)$ given by eq. (9) with a spectrometer having a resolution function $R(t-t')$ as given by eq. (3) we obviously find for the observed intensity:

$$I(t')=\frac{1}{ab\pi}\int_{-\infty}^{+\infty} e^{-\frac{(t'-t)^2}{a^2}}\,dt\int_{t_0-t}^{\infty} e^{-\frac{x^2}{b^2}}\,dx \tag{10}$$

where $x=t''-t$. As is easily shown, this integral may be transformed so that the observed intensity results from:

$$I(t')=\frac{1}{\sqrt{\pi(a^2+b^2)}}\int_{-\infty}^{t'} e^{-\frac{(t-t_0)^2}{a^2+b^2}}\,dt \tag{11}$$

But this means that the full width at half maximum of the observed gaussian line $\left(\text{which is } \frac{dI(t')}{dt'}\right)$ is proportional to $\sqrt{a^2+b^2}$. By using the definitions of a and b we find

$$(\delta t)_{\text{obs}}=\sqrt{(\delta t)_0^2+(\delta t)_1^2}, \tag{12}$$

which is just the well known addition theorem for gaussian functions. For such a spectrum as $f(t)$ given by eq. (9) and reproduced by our apparatus as given by eq. (11) we may still use the procedure outlined in the preceding section to find $(\delta t)_{\text{obs}}$, and from that we may calculate the line shape defined by $(\delta t)_1$, if $(\delta t)_0$ is known. Also the point of inflection of $I(t')$ defines t_0 exactly.

When the spectrum $f(t)$ is not given by eq. (9) but has a triangular shape rounded off by a gaussian shaped front the same conclusions remain valid, except that the point of inflection now gives a value t_0' which should be corrected as given in Fig. 4 to give the true "cut-off" position t_0.

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